

## Lecture 17

April 15<sup>th</sup>, 2004

### Higher boundary regularity

We extend our results to include the boundary.

*Higher a priori regularity upto the boundary.*     Let  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$  be a solution of

$$\begin{array}{llll} Lu & = & f & \text{on } \Omega, \\ u & = & \varphi & \text{on } \partial\Omega. \end{array}$$

Assume uniformly elliptic and that all coefficients are in  $\mathcal{C}^{k,\alpha}(\bar{\Omega})$  with  $0 < \alpha < 1$  and that  $\Omega$  is a  $\mathcal{C}^{k+2,\alpha}$  domain. If  $f \in \mathcal{C}^{k,\alpha}(\bar{\Omega})$  and  $\varphi \in \mathcal{C}^{k+2,\alpha}(\partial\Omega)$  then  $u \in \mathcal{C}^{k+2,\alpha}$ .

*Proof.* For  $k = 0$  our previous results apply unchanged (the case  $c \not\leq 0$  can be handled if one believes the Remark above).

$k = 1$ , the crucial case, we use once again difference quotients. As usual, localize to  $B^+ := B(x_0, R) \cap \bar{\Omega}$ ,  $x_0 \in \partial\Omega$ . Then flatten the boundary with the help of a  $\mathcal{C}^{3,\alpha}$  diffeomorphism  $\Psi$ . Assume the flat portion lies on the  $x_n = 0$  hyperplane. We get

$$\tilde{L}\Delta^h \tilde{u} = \Delta^h \tilde{f} - \Delta^h \tilde{a}^{ij} \cdot D_{ij} \tilde{u}^h - \Delta^h \tilde{b}_i \cdot D_i \tilde{u}^h - \Delta^h \tilde{c} \cdot \tilde{u}^h.$$

We know the RHS is uniformly  $\mathcal{C}^\alpha(\Psi(B^+))$  bounded, while the LHS is only so for the directions  $l = 1, \dots, n-1$ , the tangent directions on  $\Psi(\partial B^+)$ , since the equation  $u = \varphi$  holds there and may be differentiated in those directions (and  $\varphi$  has 3 derivatives).

We therefore use Schauder estimates for  $\Delta^h \tilde{u}$  which give it is uniformly bounded in  $\mathcal{C}^{2,\alpha}(\Psi(B^+ \iota))$ ,  $\forall B^+ \iota \subseteq B^+$  similarly to the higher regularity Theorem for the interior. This is so since the estimates used there hold, in fact, upto the boundary. We get therefore  $D_l \tilde{u} \in \mathcal{C}^{2,\alpha}(\Psi(B^+))$ ,  $l = 1, \dots, n-1$ .

We treat the remaining derivative. We have  $D_i D_l \tilde{u} \in \mathcal{C}^{1,\alpha}(\tilde{B}^+)$ ,  $i = 1, \dots, n$ ,  $l = 1, \dots, n-1 \Leftrightarrow D_{nl} \tilde{u} = D_l(D_n \tilde{u})$  (mixed derivatives commute as  $\tilde{u} \in \mathcal{C}^2$ ). So all we have to show now is  $D_{nn} \tilde{u} \in \mathcal{C}^{1,\alpha}(\tilde{B}^+)$ .

From  $\tilde{L}\tilde{u} = \tilde{f}$  we find

$$D_{nn} \tilde{u} = \frac{1}{\tilde{a}^{nn}} (\tilde{f} - (\tilde{L} - \tilde{a}^{nn})\tilde{u}).$$

From previous calculations of the form of  $\tilde{a}$  we see that choosing  $\Psi$  appropriately we may diagonalize it. Then uniformly elliptic gives  $\frac{1}{\tilde{a}^{nn}} > \gamma > 0$ . The RHS is  $\mathcal{C}^{1,\alpha}(\tilde{B}^+) \Leftrightarrow$  so is LHS  $\Leftrightarrow D\tilde{u} \in \mathcal{C}^{2,\alpha}(\Psi(B^+)) \Leftrightarrow u$  is  $\mathcal{C}^{3,\alpha}$  near  $x_0$ .

The cases  $k \geq 2$  are handled as in the interior Theorem. ■

This wraps up our discussion on Hölder spaces/norms.

## Hilbert spaces

Let  $V$  be a vector space over the field  $\mathbb{R}$ . Let  $(\cdot, \cdot)$  be a map  $V \times V \rightarrow \mathbb{R}$  such that

- i)  $(x, y) = (y, x)$
- ii)  $(\alpha_1 x_1 + \alpha_2 x, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y), \quad \forall \alpha_i \in \mathbb{R}$
- iii)  $(x, x) > 0, \quad \forall x \neq 0$

Let  $\|x\| := (x, x)^{\frac{1}{2}}$ . One can then demonstrate

$$\|(x, y)\| \leq \|x\| \cdot \|y\| \quad \text{Schwarz inequality}$$

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{triangle inequality}$$

The 2<sup>nd</sup> affirms that  $\|\cdot\|$  defines a norm.

If  $\|\cdot\|$  is complete  $(V, (\cdot, \cdot))$  is a *Hilbert space*.

Let  $F : V \rightarrow \mathbb{R}$  be linear, i.e a linear functional on  $V$ . If  $\sup_{0 \neq x \in V} \frac{|F(x)|}{\|x\|} =: \|F\|_{V^*} < \infty$ ,  $F$  is *bounded*. Here  $V^* = \{\text{bounded linear functional on } V\}$ . Similary for a Hilbert space  $\mathcal{H}$  define similarly  $\mathcal{H}^*$ .

We give the statement of the main theorem regarding Hilbert spaces . Like the Continuity Method it will serve us as a strong tool for us to attack abstract questions, a tool from Functional Analysis.

**Riesz Representation Theorem.** *Let  $\mathcal{H}$  be a Hilbert space ,  $F \in \mathcal{H}^*$ . Then  $\exists! f \in \mathcal{H}$  such that*

$$i) F(x) = (f, x), \quad \forall x \in \mathcal{H}$$

$$ii) \|F\|_{\mathcal{H}^*} = \|f\|_{\mathcal{H}}$$

$$\text{In particular} \quad \Leftrightarrow \quad \mathcal{H} = \mathcal{H}^*.$$

## Sobolev Spaces

### Motivation

If  $\Delta u = f, u \in \mathcal{C}^2(\Omega)$  then  $\forall \varphi \in \mathcal{C}_0^1(\Omega)$   $\varphi \Delta u = \varphi f$  and

$$-\int_{\Omega} \nabla \varphi \nabla u = \int_{\Omega} \Delta u \cdot \varphi = \int_{\Omega} f \cdot \varphi.$$

This observation lies at the heart of *weak* formulations of the Laplace equation.

Define an inner product on  $\mathcal{C}_0^1(\Omega) := \text{compactly supported functions in } \mathcal{C}^1(\Omega)$

$$(\varphi_1, \varphi_2) := \int_{\Omega} \nabla \varphi_1 \nabla \varphi_2.$$

$(\mathcal{C}_0^1(\Omega), (\cdot, \cdot))$  is not complete: a sequence of functions may degenerate to a function which is not everywhere differentiable though continuous. Denote by  $W_0^{1,2}(\Omega)$  the completion of  $\mathcal{C}_0^1(\Omega)$  WRT

this norm. It is nice to note that  $(\cdot, \cdot)$  extends to an inner product on  $W_0^{1,2}(\Omega)$ : represent any two elements there as limits of sequences of elements in  $\mathcal{C}_0^1(\Omega)$  and take the limit of the inner products of those, which are well defined. Hence  $W_0^{1,2}(\Omega)$  is a Hilbert space.

At this stage we do not yet know how  $W_0^{1,2}(\Omega)$  looks like. Maybe its elements are not even functions.

We continue with the motivation for defining those spaces. Let  $F(\varphi) := -\int_{\Omega} f \cdot \varphi$ ,  $\forall \varphi \in \mathcal{C}_0^1(\Omega)$ .  $F = F(f, \Omega)$  extends to a linear functional on  $W_0^{1,2}(\Omega)$ .

**Claim.** *F is bounded.*

$$\|F\| = \sup_{\substack{\varphi \in W_0^{1,2}(\Omega) = \mathcal{C}_0^1(\Omega) \\ \varphi \neq 0}} \frac{|F(\varphi)|}{\|\varphi\|_{W_0^{1,2}}} = \sup_{\substack{\varphi \in \mathcal{C}_0^1(\Omega) \\ \varphi \neq 0}} \frac{|F(\varphi)|}{\|\varphi\|_{W_0^{1,2}}}$$

since  $\mathcal{C}_0^1(\Omega)$  is dense in its completion  $W_0^{1,2}(\Omega)$ .

$$\frac{|F(\varphi)|}{\|\varphi\|_{W_0^{1,2}}} = \frac{|\int_{\Omega} \varphi \cdot f|}{\left(\int_{\Omega} |\nabla \varphi|^2\right)^{\frac{1}{2}}} \leq \frac{\left(\int_{\Omega} \varphi^2\right)^{\frac{1}{2}} \cdot \left(\int_{\Omega} f^2\right)^{\frac{1}{2}}}{\left(\int_{\Omega} |\nabla \varphi|^2\right)^{\frac{1}{2}}}.$$

Using the Poincaré inequality  $\int_{\Omega} \varphi^2 \leq c(\Omega) \cdot \int_{\Omega} |\nabla \varphi|^2$  we find a bound depending on  $\Omega, f$  but not on  $\varphi$ . ■

Hence by the Riesz Representation Theorem exists  $u \in W_0^{1,2}(\Omega)$ , though we do not know it is a function or even if so whether it has any regularity, such that

$$\begin{array}{ccc} F(\varphi) = (u, \varphi) & & \\ \text{by def.} & \quad & \text{by def.} \\ \parallel & & \parallel \\ -\int_{\Omega} f \cdot \varphi & & \int_{\Omega} \nabla u \nabla \varphi. \end{array}$$

We do not know if  $u \in C_0^1(\Omega)$ , just that  $u \in W_0^{1,2}(\Omega)$ . We have a weak formulation of

$$\begin{array}{rclcl} \Delta u & = & f & \text{on} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega. \end{array}$$

for any  $f \in L^2(\Omega)$ ! Our plan is now: if  $f$  has certain regularity,  $u$  has that regularity +2. The philosophy is instead of classically solving the  $\Delta$ -equation with an exact explicit solution like Poisson's Integral Formula etc. we just enlarge our function spaces. Then the existence of a solution in the enlarged space becomes trivial (following Riesz). The work comes down to showing that the solution actually lies back in our original space of functions! That is regularity theory in a nutshell. We will focus on that in the sequel.

## Weak derivatives

For  $u, v_i \in L_{\text{loc}}^1(\Omega)$  say " $v_i = D_i u$ " if

$$\int_{\Omega} v_i \varphi = - \int_{\Omega} u \cdot D_i \varphi, \quad \forall \varphi \in C_0^1(\Omega).$$

If such  $v$  exists  $\forall i = 1, \dots, n$  then  $u$  is *weakly differentiable* in  $\Omega$  with  $\nabla u =^{\text{weak}} (v_1, \dots, v_n)$ .

If each  $D_j u$  satisfies the above conditions we say  $u$  is twice weakly differentiable. We will omit the quotations marks in what follows.

The derivative does not exist pointwise in general. But by the Lebesgue's Theorem it does exist pointwise almost everywhere (a.e.).

## Definition

We are now in a position to define Sobolev spaces. Let  $\|u\|_{L^p(\Omega)} := \left( \int_{\Omega} |u|^p \right)^{\frac{1}{p}}$ . Define

$$L^p(\Omega) := \{\text{equivalence classes of measurable functions such that } \|\cdot\|_{L^p(\Omega)} < \infty\}$$

where  $f \sim g$  if  $f = g$  a.e.

Define

$$W^k(\Omega) := \{k\text{-times weakly differentiable functions}\} \cap L^1_{\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega),$$

Similarly define the *Sobolev spaces*

$$W^{k,p}(\Omega) \equiv L^{k,p}(\Omega) = \{u \in W^k(\Omega), D^\alpha u \in L^p(\Omega) \mid \text{all multi-indices } \alpha, |\alpha| \leq k\} \subseteq L^1_{\text{loc}}(\Omega),$$

equipped with the norm

$$\|\cdot\|_{W^{k,p}(\Omega)} := \left\{ \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha \cdot|^p \right\}^{\frac{1}{p}}$$

(still need to prove it is a norm!). An equivalent norm is given by

$$\sum_{|\alpha| \leq k} \int_{\Omega} \|D^\alpha \cdot\|_{L^p(\Omega)}.$$

$L^p(\Omega)$  is a Banach space ! (Riesz-Fischer Theorem). Also  $W^{k,p}(\Omega) = L^{k,p}(\Omega)$  are.

**Claim.**  $\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$  is dense in  $W^{k,p}(\Omega)$ . i.e. we could have defined  $W^{k,p}(\Omega)$  as the completion of  $\mathcal{C}^\infty(\Omega)$  WRT  $\|\cdot\|_{W^{k,p}(\Omega)}$ .

Given  $u \in W^{k,p}(\Omega)$  mollify it to

$$u_h(x) := \int_{\mathbb{R}^n} \frac{1}{h^n} \rho\left(\frac{|x-y|}{h}\right) u(y) dy,$$

with  $\rho$  a smooth bump function on  $\mathbb{R}$  with mass 1 and support in  $[-\frac{1}{2}, \frac{1}{2}]$ . Now  $u \in \mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)$

and  $u_h \rightarrow u$  in the  $W^{k,p}(\Omega)$  norm. ■

We now define Sobolev spaces of compactly supported objects

$$W_0^{k,p}(\Omega) := \text{completion of } \mathcal{C}_0^k(\Omega) \text{ WRT } \|\cdot\|_{W^{k,p}(\Omega)}.$$

Note functions in  $\mathcal{C}_0^k(\Omega)$  vanish on  $\partial\Omega$  so in a sense  $W_0^{1,p}(\Omega)$  (respectively  $W_0^{k,p}(\Omega)$ ) can be thought of as (weak) functions which vanish on  $\partial\Omega$  (whose first  $k - 1$  derivatives vanish on  $\partial\Omega$ ).

## Equivalence of norms

For  $\varphi \in W_0^{1,2}(\Omega)$  we defined two norms. One using the inner product  $\int_{\Omega} \nabla \varphi_1 \cdot \nabla \varphi_2$  on  $\mathcal{C}_0^1(\Omega)$  which gave us the norm

$$\|\varphi\| = \left\{ \int_{\Omega} |\nabla \varphi|^2 \right\}^{\frac{1}{2}}$$

and another norm

$$\|\varphi\|' = \left\{ \sum_{|\alpha| \leq 1} \int_{\Omega} |D^{\alpha} \varphi|^2 \right\}^{\frac{1}{2}} = \left\{ \int_{\Omega} |\varphi|^2 + \sum_{i=1}^n |D^i \varphi|^2 \right\}^{\frac{1}{2}} \leq \|\varphi\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)}.$$

These norms are indeed equivalent since we are assuming compact support! The Poincaré inequality shows  $\|\cdot\|' \leq (1 + c(\Omega)) \cdot \|\cdot\|$ . This inequality fails grossly for non-compactly supported functions, e.g the constant function. Since  $\|\cdot\| \leq \|\cdot\|'$  the norms are equivalent.

**Remark.** in both of the above norms we define first the norms of functions which are also in  $\mathcal{C}_0^1(\Omega)$  and then we extend the norm to the completion by means of norms of limits of sequences whose elements are all in  $\mathcal{C}_0^1(\Omega)$  (those are dense).